Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Thursday 11 June 2020

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **9FM0/3C**

Further Mathematics

Advanced

Paper 3C: Further Mechanics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Unless otherwise indicated, whenever a numerical value of g is required, take $g = 9.8\,\text{m}\,\text{s}^{-2}$ and give your answer to either 2 significant figures or 3 significant figures.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

P66507A

1/1/1/1/

P 6 6 5 0 7 A 0 1 2 8

Pearson

1. A particle P of mass 0.5 kg is moving with velocity $(4\mathbf{i} + 3\mathbf{j})$ m s^{-1} when it receives an impulse $\mathbf{J}$ N s. Immediately after receiving the impulse, P is moving with velocity $(-\mathbf{i} + 6\mathbf{j})$ m s^{-1}.

(a) Find the magnitude of $\mathbf{J}$.

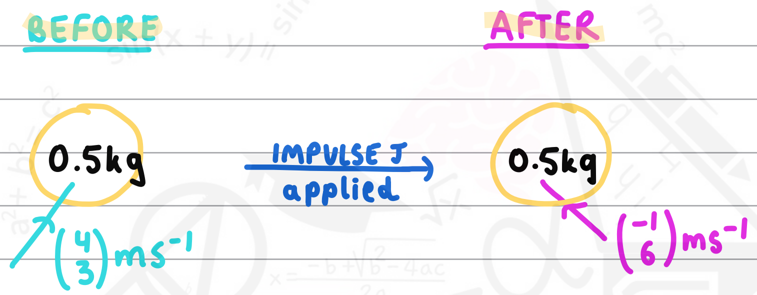
(4)

The angle between the direction of the impulse and the direction of motion of P immediately before receiving the impulse is $\alpha°$

(b) Find the value of α

(3)

(a) illustrating the above - KEEPING the velocity direction



∴ need to find the IMPULSE that had caused the particle to change direction

...using vector form of the Impulse-momentum principle:

$$I = m(\underline{v} - \underline{u})$$

subbing in:

$$J = 0.5\left(\begin{pmatrix}-1\\6\end{pmatrix} - \begin{pmatrix}4\\3\end{pmatrix}\right)$$

$$\Rightarrow J = 0.5\begin{pmatrix}-5\\3\end{pmatrix}$$

factoring 0.5 into the bracket

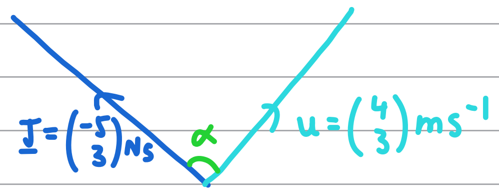
$$\Rightarrow J = \begin{pmatrix}-2.5\\1.5\end{pmatrix}$$

now finding the magnitude of J requires us to Pythagorise it:

$$\boxed{|J| = \sqrt{(-2.5)^2 + (1.5)^2} = \frac{\sqrt{34}}{2} \text{ Ns}}$$

(b) sketching the correct angle 'α' needed - i.e the one between the direction of the impulse (NOTE: can just be any scalar multiple of it, so $\begin{pmatrix}-5\\3\end{pmatrix}$ is fine and will make our arithmetic easier than if we were dealing with decimals) and the velocity before it :

Question 1 continued



} to find this, two main methods:

METHOD 1: using the formula for angle between two vectors

formula: $\cos\theta = \dfrac{a.b}{|a||b|}$ ← scalar product, ← product of magnitudes

-subbing into this: $\cos\theta = \dfrac{\underline{I}\cdot\underline{u}}{|\underline{I}||\underline{u}|}$

$$\Rightarrow \cos\theta = \frac{\begin{pmatrix}-5\\3\end{pmatrix}\cdot\begin{pmatrix}4\\3\end{pmatrix}}{\sqrt{(-5)^2+(3)^2}\sqrt{(4)^2+(3)^2}} = \frac{-5(4)+3(3)}{\sqrt{25}\sqrt{34}} = \frac{-11}{5\sqrt{34}}$$

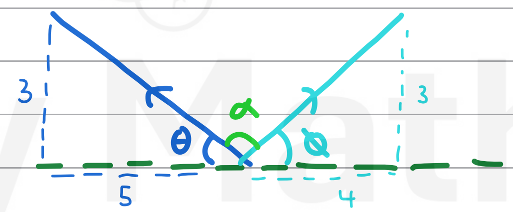
$$\therefore \theta = \cos^{-1}\left(\frac{-11}{5\sqrt{34}}\right)$$

$$= 112.16634\ldots$$

$$\boxed{= 112^\circ \text{ (3 s.f)}}$$

METHOD 2: using properties of straight lines and trig

manipulating previous vector diagram to exploit straight angle properties
-let impulse make angle θ to the straight line and the velocity before make angle ϕ to the straight line



hence, the required angle =
$180^\circ - \tan^{-1}(3/5) - \tan^{-1}(3/4)$

$= 112.166345\ldots$

$\boxed{= 112^\circ \text{ (3 s.f)}}$

(Total for Question 1 is 7 marks)

2. A truck of mass 1200 kg is moving along a straight horizontal road.

At the instant when the speed of the truck is $v\,\mathrm{m\,s^{-1}}$, the resistance to the motion of the truck is modelled as a force of magnitude $(900 + 9v)$ N.

The engine of the truck is working at a constant rate of 25 kW.

(a) Find the deceleration of the truck at the instant when $v = 25$

(4)

Later on, the truck is moving up a straight road that is inclined at an angle θ to the horizontal, where $\sin\theta = \frac{1}{20}$

At the instant when the speed of the truck is $v\,\mathrm{m\,s^{-1}}$, the resistance to the motion of the truck from non-gravitational forces is modelled as a force of magnitude $(900 + 9v)$ N.

When the engine of the truck is working at a constant rate of 25 kW the truck is moving up the road at a constant speed of $V\,\mathrm{m\,s^{-1}}$.

(b) Find the value of V.

(5)

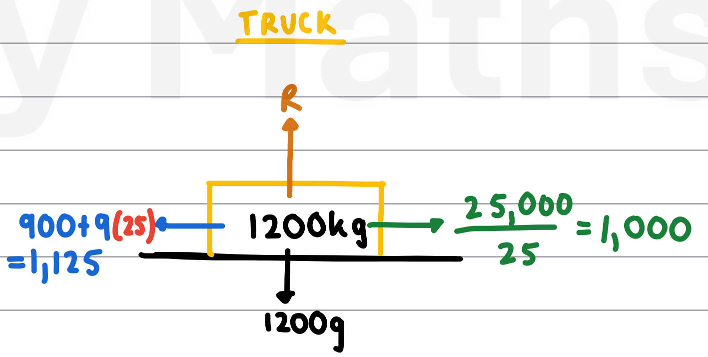
(a) let's illustrate the above information on a detailed force diagram
↳ label the resistance, the REACTION FORCE and the POWER rearranged:

NOTE: could've calculated this power as a separate line of working but much more efficient in exam to just calculate straight onto diagram

formula: $P = Fv$
POWER in Watts, FORCE in Newtons, VELOCITY in $\mathrm{ms^{-1}}$

$\Rightarrow F = \frac{P}{v}$

convert into Watts
$25\,\mathrm{kW} \xrightarrow{\times 1000} 25{,}000\,\mathrm{W}$
and $v = 25$



the question is asking us to find the deceleration of the car - know from Chp 2 FM1 or Chp 5 Mechanics Yr 2 - this would require us to resolve parallel to the plane

$R(\rightarrow):\ 1{,}000 - 1{,}125 = 1{,}200a$
$\Rightarrow -125 = 1{,}200a$

Question 2 continued

$$\div 1{,}200 \qquad \div 1{,}200$$

$$\Rightarrow a = -\frac{125}{1{,}200} = -\frac{5}{48}\ ms^{-1}$$

we know that a -ve acceleration counts as a deceleration $\therefore$ truck's deceleration = $\boxed{\frac{5}{48}\ ms^{-2}}$

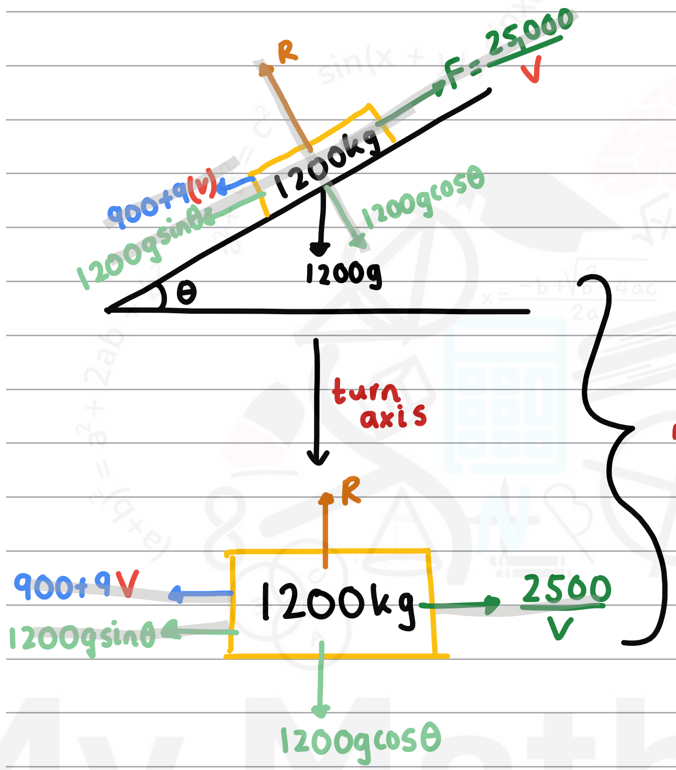
(b) let's look at the forces again - redrawing the part (a) diagram but on an inclined plane - label the resistance, the reaction force (perpendicular to the surface of impact) and the power rearranged (like in part (a)):

$P = FV$

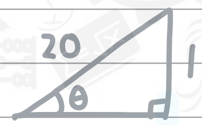
POWER in Watts; FORCE in Newtons; VELOCITY in ms^{-1}

$\Rightarrow F = \frac{P}{V}$

where $P = 25{,}000$ W and $v = V$



where if $\sin\theta = 1/20$ - drawing appropriate right-angled triangle and using Pythagoras' rearranged:



$(20)^2 - (1)^2 = x^2$

$400 - 1 = x^2$

$\Rightarrow x^2 = 399$

$\Rightarrow x = \sqrt{399}$

$\cos\theta = A/H = \sqrt{399}/20$

$\tan\theta = O/A = 1/\sqrt{399}$

now, the fact that the truck is now moving at a constant speed of V implies that $a = 0$ - hence bearing in mind Newton's Second Law of Motion -

WAY 1: using memorised 'forces left = forces right'	WAY 2: subbing into $\Sigma F = ma$
$R(\rightarrow)$:	$R(\rightarrow)$:
$\Rightarrow 900 + 9(V) + 1200g\sin\theta = \frac{25{,}000}{V}$	$900 + 9V + 1200g\sin\theta - \frac{25{,}000}{V} = 1200(0)$
	$\Rightarrow 900 + 9v + 1200g\sin\theta = \frac{25{,}000}{V}$

Question 2 continued

know $\sin\theta = 1/20$, hence subbing it in

$$900 + 9v + 1200g\left(\frac{1}{20}\right) = \frac{25{,}000}{v}$$

expand brackets

$$900 + 9v + 60g = \frac{25{,}000}{v}$$

$\times v$ $\qquad$ $\times v$

$$9v^2 + v(900 + 60g) - 25{,}000 = 0$$

$$\Rightarrow 9v^2 + v(900 + 60(9.8)) - 25{,}000 = 0$$

$$\Rightarrow 9v^2 + 1488v - 25{,}000 = 0$$

solve using calc equtn solver

$v = 15.37187\ldots, -180.7025\ldots$

but $v > 0$ (doesn't change direction)

$$\boxed{\therefore v = 15.4\,\text{ms}^{-1}\ (3\text{ s.f})}$$

Question 2 continued

(Total for Question 2 is 9 marks)

3. Two particles, A and B, have masses $3m$ and $4m$ respectively. The particles are moving in the same direction along the same straight line on a smooth horizontal surface when they collide directly. Immediately before the collision the speed of A is $2u$ and the speed of B is u.

The coefficient of restitution between A and B is e.

(a) Show that the direction of motion of each of the particles is unchanged by the collision.

(8)

After the collision with A, particle B collides directly with a third particle, C, of mass $2m$, which is at rest on the surface.

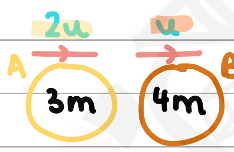
The coefficient of restitution between B and C is also e.

(b) Show that there will be a second collision between A and B.

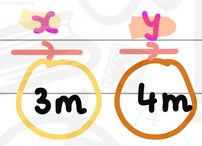
(6)

(a) illustrating this elastic collision in 1D diagrammatically - label the respective speeds, direction of motion, etc.

BEFORE:



AFTER:



NOTE: by modelling the velocities IN THIS WAY, as in their velocity direction AFTER are unchanged - the final aim of the question is to show that $x, y > 0$

following the usual procedure for elastic collisions in 1D - notice how both speeds after are unknown ∴ can't just stop at using PCLM - need to do NEL (Impact law) as well

...first PCLM - means the total momentum before the collision equals the total momentum after:

formula: $m_A u_A + m_B u_B = m_A v_A + m_B v_B$

sub into above

$3\cancel{m}(2u) + 4\cancel{m}(u) = 3\cancel{m}(x) + 4\cancel{m}(y)$

cancel m's, then expand brackets

$\Rightarrow 3x + 4y = 6u + 4u$

$\Rightarrow 3x + 4y = 10u$ - ①

...next, NEL - i.e the formula to find the coefficient of restitution:

$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{v_Q - v_P}{u_P - u_Q}$$

subbing into above

Question 3 continued

$e = \frac{y-x}{2u-u} = \frac{y-x}{u}$

$\times u \qquad \times u$

$\Rightarrow y - x = eu$ -②

solve ① and ② simultaneously - first eliminate 'x':

① + 3×②

$3x + 4y = 10u$

$+$ $-3x + 3y = 3eu$

$7y = 10u + 3eu$

factorise the u's on the RHS

$7y = u(10 + 3e)$

$\div 7 \qquad \div 7$

$\boxed{\Rightarrow y = \frac{u}{7}(10 + 3e)}$

- now, eliminate 'y':

① − 4×②

$3x + 4y = 10u$

$-$ $-4x + 4y = 4eu$

$7x = 10u - 4eu$

factorise 'u' out:

$7x = u(10 - 4e)$

$\div 7 \qquad \div 7$

$\boxed{x = \frac{u}{7}(10 - 4e)}$

and finally proving $x, y > 0$: mainly using $0 < e \leq 1$

... x :

if $0 < e \leq 1$,

$\Rightarrow 10 - 4e > 0$

... y :

if $0 < e \leq 1$,

$\Rightarrow 10 + 3e > 0$

$\therefore$ both particles are travelling in same direction after as before collision $\boxed{\Rightarrow \text{unchanged}}$

(b) now the elastic collisions in 1D question turns into a SUCCESSIVE COLLISIONS one that involves three particles... a couple of things to remember:

- clear, descriptive, labelled diagram that contains all three particles
- consistent system of labelling } here, coloured: u, v = before first collision
WAY I labelling: x, y = after first collision
p, q = after second collision

Question 3 continued

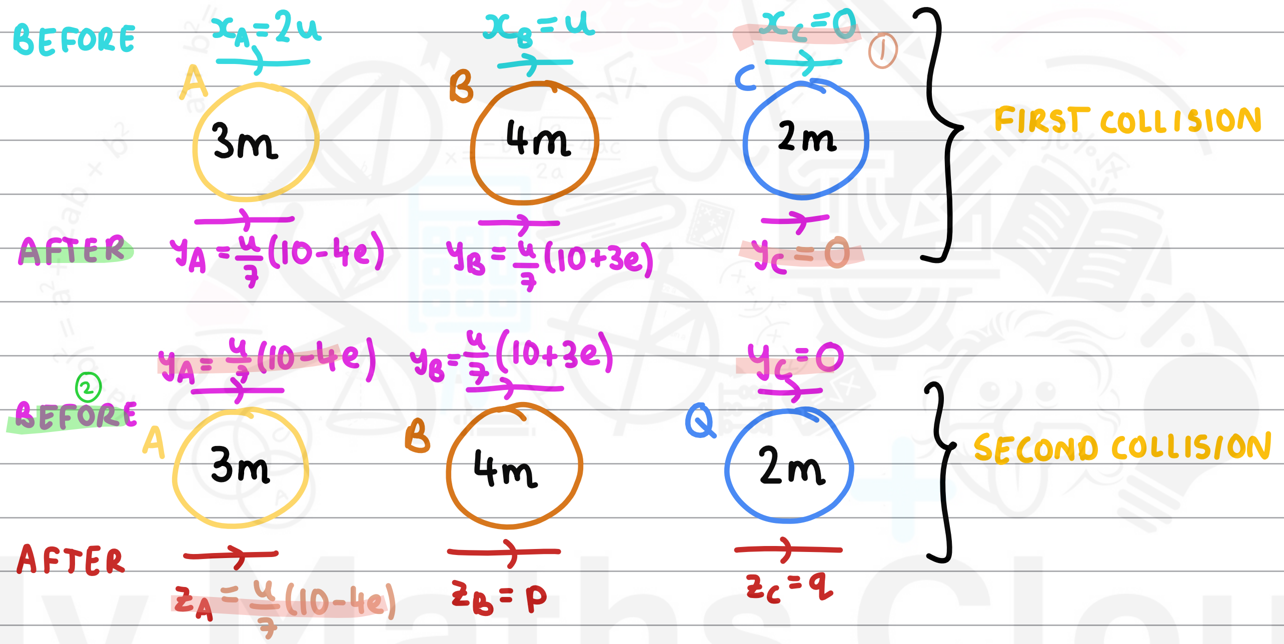
OR in exam, WAY 2 labelling: x_A, x_B, x_C = before first collision
y_A, y_B, y_C = after first collision
z_A, z_B, z_C = after second collision

↳ NOTE: in theory, doesn't matter that this is inconsistent with part (a), but I personally find it easier to solve equtns with different letters of the alphabet (WAY 1) rather than with x_A and x_B (WAY 2)!

- always look at what variables do not change - i.e if the particle A is not involved in a collision, then $u_A = v_A$ ①

- in general, $v_{\text{PARTICLE FIRST COLLISION}} = u_{\text{PARTICLE SECOND COLLISION}}$

...bearing these in mind, adding to the diagram from (a)



now, for there to be a third collision (i.e for B to collide with A again), know that $z_A = \frac{4}{7}(10-4e) > z_B = p$ ⟹ need to find z_B (i.e p)

following the usual procedure for collisions in 1D between B and C (second collision) - see both z_B and z_C are unknown ∴ not only need PCLM but NEL (Impact Law) as well

...first, PCLM:

formula: $m_B u_B + m_C u_C = m_B v_B + m_C v_C$

subbing into above:

P 6 6 5 0 7 A 0 1 0 2 8

NOTE: keep as y_B for now to simplify our arithmetic - sub in expression for it later

$$4\cancel{m}(y_B) + 2\cancel{m}(0) = 4\cancel{m}(p) + 2\cancel{m}(q)$$

$$\Rightarrow 4y_B = 4p + 2q$$

$$\Rightarrow 4p + 2q = 4y_B \quad -①$$

...next NEL:

$$e = \frac{q-p}{y_B - 0} \quad \Rightarrow q - p = ey_B \quad -②$$

solving ① and ② simultaneously - eliminate 'q'

①$-2\times$②

$$\begin{aligned} 4p + 2q &= 4y_B \\ -\;\; -2p + 2q &= 2ey_B \\ \hline 6p &= 4y_B - 2ey_B \end{aligned}$$

$\div 2$ $\qquad$ $\div 2$

$$3p = 2y_B - ey_B$$

factorise y_B

$$3p = y_B(2-e)$$

$\div 3$ $\qquad$ $\div 3$

$$p = \frac{y_B}{3}(2-e)$$

Now sub in $y_B = \frac{4}{7}(10+3e)$

$$\therefore p = \frac{\frac{4}{7}(10+3e)}{3}(2-e)$$

$$\Rightarrow p = \frac{4}{21}(10+3e)(2-e)$$

now need to prove $z_A > z_B$ - using proof by deduction

conjecture: $\frac{\cancel{4}}{7}(10-4e) > \frac{\cancel{4}}{21}(10+3e)(2-e)$

cancel 4's and expand double brackets

$$\frac{1}{7}(10-4e) > \frac{1}{21}(20-4e-3e^2)$$

$\times 21$ $\qquad$ $\times 21$

$$3(10-4e) > 20-4e-3e^2$$

expand

$$30-12e > 20-4e-3e^2$$

collect like terms

$$3e^2 - 8e + 10 > 0$$

to prove - complete square

$\div 3$ $\qquad$ $\div 3$

$$e^2 - \frac{8}{3}e + \frac{10}{3} > 0$$

complete the square

$$\left(e - \frac{4}{3}\right)^2 - \frac{16}{9} + \frac{10}{3} > 0$$

$$\Rightarrow \left(e - 4/3\right)^2 + \frac{14}{9} > 0$$

Question 3 continued

through the trivial inequality,

$$(e - 4/3)^2 > 0$$

theorem:

$$\Rightarrow (e - 4/3)^2 + 14/9 > 14/9$$

always +ve ∴ our conjecture, that $z_A > z_B$ must be true

$\boxed{\text{∴ there IS a second collision between A and B}}$

(Total for Question 3 is 14 marks)

4. [*In this question,* **i** *and* **j** *are perpendicular unit vectors in a horizontal plane.*]

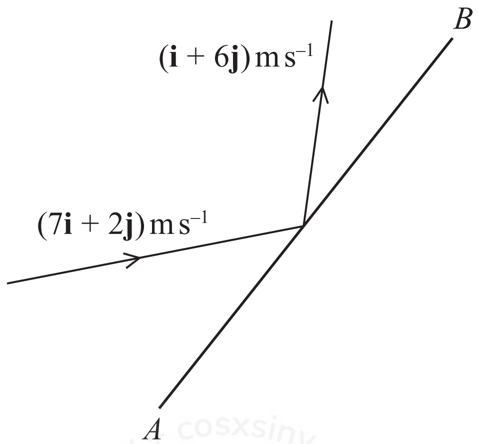


Figure 1

Figure 1 represents the plan view of part of a smooth horizontal floor, where AB represents a fixed smooth vertical wall.

A small ball of mass 0.5 kg is moving on the floor when it strikes the wall.

Immediately before the impact the velocity of the ball is $(7\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$.

Immediately after the impact the velocity of the ball is $(\mathbf{i} + 6\mathbf{j})\,\text{m s}^{-1}$.

The coefficient of restitution between the ball and the wall is e.

(a) Show that AB is parallel to $(2\mathbf{i} + 3\mathbf{j})$.

(4)

(b) Find the value of e.

(5)

(a) recognising this as an elastic collisions in 2D question but no fixed vertical wall - hence not surprised that the question asks for a vector for the wall AB :

...couple of ways to do this:

METHOD 1: using impulse-momentum principle

using the fact that we know that in elastic collisions the IMPULSE always acts perpendicular to the surface of contact - hence if we found the IMPULSE through the impulse-momentum principle - the vector for the wall AB would just be the vector perpendicular to the IMPULSE

subbing our 'before' and 'after' velocities into impulse momentum principle:

$\underline{I} = m(\underline{v} - \underline{u})$

Question 4 continued

$$I = 0.5\left(\begin{pmatrix}1\\6\end{pmatrix}-\begin{pmatrix}7\\2\end{pmatrix}\right)$$

$$= 0.5\begin{pmatrix}-6\\4\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}$$

∴ need a vector perpendicular to the IMPULSE

WAY 1: using dot product by inspection:

know that for the wall vector - call it $\begin{pmatrix}a\\b\end{pmatrix}$ to be perpendicular to 'I', need:

$$I \cdot \begin{pmatrix}a\\b\end{pmatrix} = 0$$

$$\begin{pmatrix}-3\\2\end{pmatrix}\cdot\begin{pmatrix}a\\b\end{pmatrix} = 0$$

∴ by INSPECTION (know need to use same digits as impulse: $\boxed{\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}}$

WAY 2: turning vectors into linear equations and exploit perp. lines properties

$I = \begin{pmatrix}-3\\2\end{pmatrix}$ - could interpret this as the gradient of a line:

$$y = -\tfrac{3}{2}x$$

$$m_1 = -\tfrac{3}{2}$$

∴ using $m_1 \times m_2 = -1$ for lines perpendicular to each other

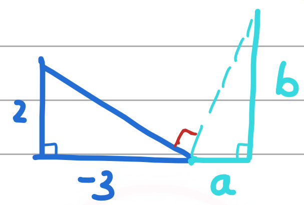
$$m_2 = \tfrac{2}{3}$$

∴ as a LINE

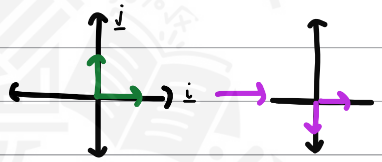
$$\Rightarrow y = \tfrac{2}{3}x$$

$\boxed{\therefore \text{vector wall} = \begin{pmatrix}2\\3\end{pmatrix}}$

WAY 3: think of it as a linear transformations question - involves MATRICES



know that a rotation 90° clockwise is represented by:



$\begin{pmatrix}1&0\\0&1\end{pmatrix}$ → $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

∴ rotating $\begin{pmatrix}-3\\2\end{pmatrix}$ 90° clockwise (follow $Mx = y$)

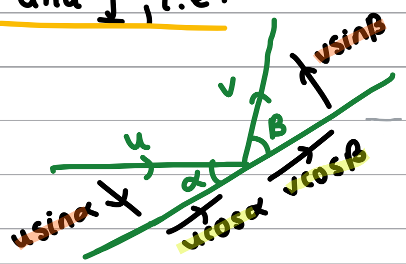
$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}-3\\2\end{pmatrix} =$$

MATRIX MULTIPLICATION 'rows into columns'

$$\begin{pmatrix}0(-3)+1(2)\\-1(-3)+0(2)\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

$\boxed{\therefore \text{vector wall is } \begin{pmatrix}2\\3\end{pmatrix}}$

METHOD 2: using the 2 scalar product results for when walls are not parallel to i and j, i.e:



... parallel components:

formula: $u \cdot W = v \cdot W$

... perpendicular components:

formula: $-eu \cdot I = v \cdot I$

Question 4 continued

let the vector wall $= \begin{pmatrix} a \\ b \end{pmatrix}$

→ using the first formula (parallel components):

$$u \cdot w = v \cdot w$$

$$\begin{pmatrix} 7 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow 7a + 2b = a + 6b$$

collect like terms

$$6a = 4b$$

WAY 1: using the ratio between 'a' and 'b'

$$6a = 4b$$

$\div b \div 6$

$$\frac{a}{b} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore a : b = 2 : 3$$

$$\Rightarrow \boxed{w = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

WAY 2: using algebra

$\div 4 \qquad \div 4$

$$b = \frac{6}{4}a = \frac{3}{2}a$$

let $a = 1$

$$w = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$$

$\times 2 \qquad \times 2$

$$\boxed{w = \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

(b) METHOD 1: using the perp. component formula

now that we know $I = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ (either if ctd. to use METHOD 1 or, if METHOD 2, then I is the perp. vector to $w = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$) - can sub into perp. component formula

$$-eu \cdot I = v \cdot I$$

$$-e\begin{pmatrix} 7 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\Rightarrow -e(7(-3) + 2(2)) = 1(-3) + 6(2)$$

expand

$$-e(-21 + 4) = -3 + 12$$

$$-e(-17) = 9$$

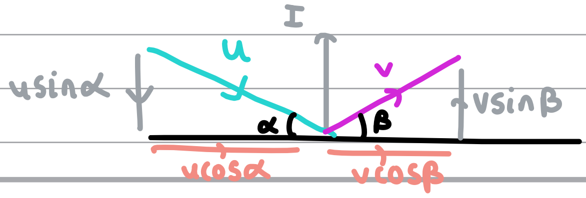
$$\therefore 17e = 9$$

$\div 17 \qquad \div 17$

$$\Rightarrow \boxed{e = 9/17}$$

METHOD 2: non-formula method: using standard oblique collisions facts

... we know from standard oblique collisions questions (NOT involving vector wall) that:

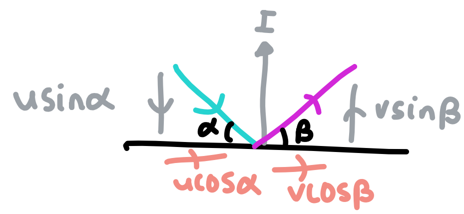


...focusing on the perp. component:

Impulse only acts perpendicular to the fixed surface so the only component that changes must be the one perp. to the surface-impacted by NEL (Impact law)

$\Rightarrow e u \sin\alpha = v \sin\beta$

↳ showing on diagram:

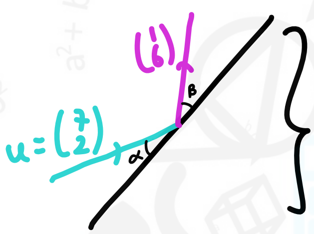


...and on the parallel component:

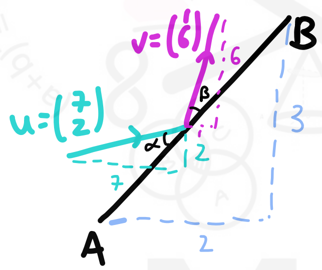
no impact 'along' the surface, hence:

$u\cos\alpha = v\cos\beta$

now we can apply the same formulae to our vector-wall situation - that involves 'e'

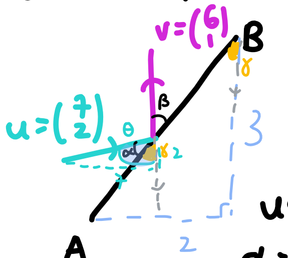


but this would require us to find 'α' and 'β' - do so using vectors for u, v and w and appropriate trig



...first need 'α':

using the blue triangle angle - call it 'θ', and subtracting the yellow angle (call it 'γ') from it - happens to be the same size as the TOP ANGLE from vector wall triangle due to corresponding angles being equal (can see if draw vertical parallel lines)



$\Big\} \alpha = \theta - \gamma$

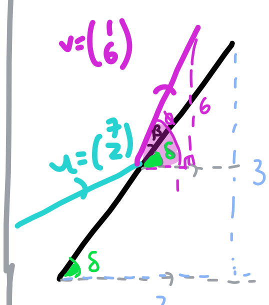
using trig

$\alpha = \tan^{-1}(7/2) - \tan^{-1}\left(\frac{2}{3}\right)$

$= 40.36...$

...next, need 'β':

using pink triangle angle - call it 'φ' and subtracting the angle δ from it - draw a horizontal parallel line - see happens to be corresponding to the bottom angle of the vector wall triangle ∴ same angle



$\beta = \phi - \delta$

$= \tan^{-1}(6/1) - \tan^{-1}(3/2) =$

$24.22..$

Question 4 continued

... now that we know 'α' and 'β', could find 'e' in two ways:

WAY 1: ctd from prev diagram and taking tan of β:

$$\tan\beta = \frac{eu\sin\alpha}{u\cos\alpha}$$

$$\tan\beta = e\tan\alpha$$

WAY 2: component form

... know parallel:

$$u\cos\alpha = v\cos\beta \quad ①$$

... and perp.

$$eu\sin\alpha = v\sin\beta \quad ②$$

② ÷ ①

$$\frac{e\cancel{u}\sin\alpha}{\cancel{u}\cos\alpha} = \frac{\cancel{v}\sin\beta}{\cancel{v}\cos\beta}$$

$$\Rightarrow e\tan\alpha = \tan\beta$$

$$\Rightarrow \tan\beta = e\tan\alpha$$

Subbing in:

$$\tan(24.22\ldots) = e\tan(40.36\ldots)$$

$$\Rightarrow \boxed{e = \frac{\tan(24.22\ldots)}{\tan(40.36\ldots)} = 9/17}$$

(Total for Question 4 is 9 marks)

5. A smooth uniform sphere P has mass 0.3 kg. Another smooth uniform sphere Q, with the same radius as P, has mass 0.2 kg.

The spheres are moving on a smooth horizontal surface when they collide obliquely. Immediately before the collision the velocity of P is $(4\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$ and the velocity of Q is $(-3\mathbf{i} + \mathbf{j})\,\text{m s}^{-1}$.

At the instant of collision, the line joining the centres of the spheres is parallel to $\mathbf{i}$.

The kinetic energy of Q immediately after the collision is half the kinetic energy of Q immediately before the collision.

(a) Find

(i) the velocity of P immediately after the collision,

(ii) the velocity of Q immediately after the collision,

(iii) the coefficient of restitution between P and Q,

carefully justifying your answers.

(11)

(b) Find the size of the angle through which the direction of motion of P is deflected by the collision.

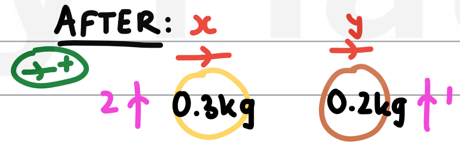
(3)

(a) notice how we have an 'oblique collisions between two spheres' question - first illustrating the collision with a diagram:

BEFORE:

split into $\underline{i}$ and $\underline{j}$ components

AFTER:



NOTE: realistically know that P might go in the opposite direction after its collision but easier to aim 'x' and 'y' RIGHTWARDS to avoid -ves in the working

...parallel components:

notice how both SPEEDS AFTER are unknown ∴ can't stop at just using PCLM - need to do NEL (Impact law) as well:

· first PCLM - means the total momentum before the collision equals the total momentum after

Question 5 continued

formula: $m_u u_p + m_Q u_Q = m_p v_p + m_Q v_Q$

$0.3(4) + 0.2(-3) = 0.3(x) + 0.2(y)$

$\Rightarrow 0.3x + 0.2y = 1.2 - 0.6$

$\Rightarrow 0.3x + 0.2y = 0.6$

$\times 10 \qquad \times 10$

$3x + 2y = 6 \quad -①$

· next, NEL - i.e formula for coefficient of restitution:

$$e = \frac{\text{speed of sep.}}{\text{speed of approach}} = \frac{v_Q - v_p}{u_p - u_Q}$$

subbing into above:

$$e = \frac{y - x}{4 - (-3)} \Rightarrow y - x = 7e \quad -②$$

but before solve this simultaneously, we also need to utilise third fact - KINETIC ENERGY

$K.E_{final} = \frac{1}{2} K.E_{initial}$

formula: $\frac{1}{2} m v^2 = \frac{1}{2} m u^2$ (mass, final velocity; mass, initial velocity)

$$\frac{1}{2}\left(\frac{1}{2}(0.2)\sqrt{(-3)^2 + (1)^2}\right)^2 = \frac{1}{2}(0.2)\left(\sqrt{y^2 + (1)^2}\right)^2$$

expand brackets

$$\frac{1}{20}(10) = \frac{1}{10}(y^2 + 1)$$

$$\frac{1}{2} = \frac{1}{10}y^2 + \frac{1}{10}$$

$$\frac{1}{10}y^2 = \frac{2}{5}$$

$\div \frac{1}{10} \qquad \div \frac{1}{10}$

$\Rightarrow y^2 = 4$

$\Rightarrow y = \pm 2$

so the E_k fact helped us find velocity after for Q -

subbing into ①

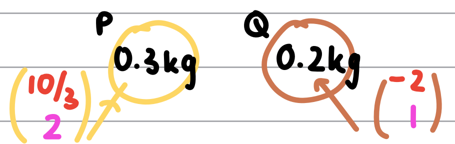
$y = -2 \qquad 3x + 2(-2) = 6$

$3x = 10$

$x = 10/3$

...checking with 'AFTER' diagram if possible:

AFTER:



but this would imply that P and Q have passed through each other (continue in SAME DIRECTION) which is IMPOSSIBLE

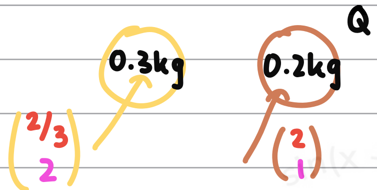
Question 5 continued

hence, left with $y=2$,

subbing into ①

$$3x+2(2)=6$$
$$\div 3 \quad 3x=2 \quad \div 3$$
$$x=\tfrac{2}{3}$$ checking with 'AFTER' diagram

AFTER:



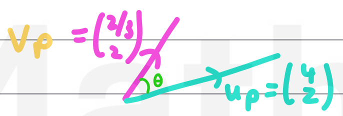
now Q changes direction ∴ PROBABLE

$$\boxed{\Rightarrow \; v_P = \begin{pmatrix} 2/3 \\ 2 \end{pmatrix} \text{ms}^{-1} \quad v_Q = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ms}^{-1}}$$

and subbing x, y into ②

$$2-\tfrac{2}{3}=7e$$
$$\frac{4}{3}=7e$$
$$\div 7 \qquad \div 7$$
$$\boxed{e=\frac{4}{21}}$$

(b) could sketch path of P to help see this angle of deflection:



METHOD 1: using angle between two lines formula

$$\frac{u_P \cdot v_P}{|u_P||v_P|} = \frac{\begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2/3 \\ 2 \end{pmatrix}}{\sqrt{(4)^2+(2)^2}\sqrt{(2/3)^2+(2)^2}}$$

scalar product

↳ product of magnitudes

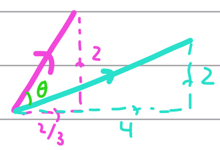
$$\cos\theta = \frac{4(2/3)+2(2)}{\sqrt{20}\sqrt{40/9}}$$

$$\Rightarrow \cos\theta = \frac{20/\cancel{3}}{\frac{20\sqrt{2}}{\cancel{3}}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

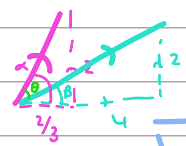
calc or trig values:

$$\boxed{\theta = 45°}$$

METHOD 2: non-formula method - using vector triangles and trig



see how the angle θ comes from angle of blue triangle, call it 'α' subtracted from pink triangle angle, call it 'β'



$$\theta = \alpha - \beta$$
$$= \tan^{-1}\left(\frac{2}{2/3}\right) - \tan^{-1}\left(\frac{2}{4}\right)$$
$$\boxed{\theta = 45°}$$

Question 5 continued

(Total for Question 5 is 14 marks)

6. A light elastic string with natural length l and modulus of elasticity kmg has one end attached to a fixed point A on a rough inclined plane. The other end of the string is attached to a package of mass m.

The plane is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{5}{12}$

The package is initially held at A. The package is then projected with speed $\sqrt{6gl}$ up a line of greatest slope of the plane and first comes to rest at the point B, where $AB = 3l$.

The coefficient of friction between the package and the plane is $\frac{1}{4}$

By modelling the package as a particle,

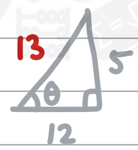
(a) show that $k = \frac{15}{26}$ **(6)**

(b) find the acceleration of the package at the instant it starts to move back down the plane from the point B. **(5)**

(a) always with 'elastic strings and springs' questions focus on drawing the correct diagram:

...label:
- string
- $\lambda = kmg$
- $l = l$

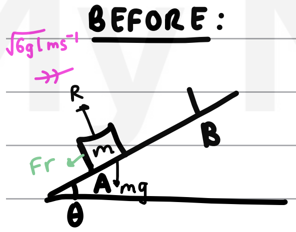
also given $\tan\theta = 5/12$, Pythag. triples: 5, 12, 13, helpful to straight away draw the angle triangle



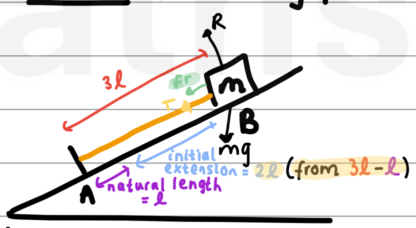
$\sin\theta = \frac{O}{H} = \frac{5}{13}$

$\cos\theta = \frac{A}{H} = \frac{12}{13}$

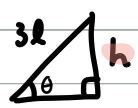
BEFORE:



AFTER (including forces)



- E.K.E (hint: 'projected')
- E.P.E (hint: 'extended')
- G.P.E → this has to be the perp. distance



$\Rightarrow \sin\theta = \frac{h}{3l}$

$\frac{5}{13} = h/3l$

$\Rightarrow h = \frac{15l}{13}$

- w.d against friction : $(Fr = \mu R) \times d$

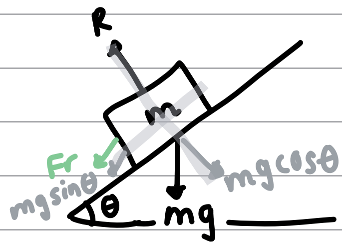
reaction force (resolve perp. to plane)

coeff. of restitution (given)

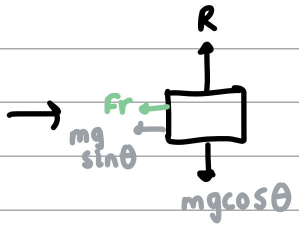
P 6 6 5 0 7 A 0 2 0 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 6 continued ...finding Fr for w.d against friction:



$\rightarrow$



$R(\updownarrow): \ R = mg\cos\theta$
$= mg(12/13)$

$Fr = \frac{1}{4}\left(\frac{\cancel{12}^{3}}{13}mg\right)$

$\Rightarrow Fr = \frac{3}{13}mg$

sub above into the work-energy principle (includes dissipative forces)

$$\underset{\text{n/a}}{w.d \text{ in}} + \underset{\text{kinetic initial}}{K.E_i} + \underset{\text{gravitational potential initial}}{G.P.E_i} + \underset{\text{elastic potential initial}}{E.P.E_i} = \underset{\text{kinetic energy final}}{K.E_f} + \underset{\text{gravitational potential final}}{G.P.E_f} + \underset{\text{elastic potential final}}{E.P.E_f} + w.d \text{ against friction}$$

$$\frac{1}{2}mu^2 + mgh_1 + \frac{\lambda x^2}{2l} = \frac{1}{2}mv^2 + mgh_2 + \frac{\lambda x^2}{2l} + Fr \times d$$

Sub into above

$$\frac{1}{2}m(\sqrt{6gl})^2 + 0 + 0 = 0 + mg\left(\frac{15l}{13}\right) + \frac{kmg(2l)^2}{2} + \frac{3}{13}mg(3l)$$

expand and cancel 'mg'

$$3\cancel{mg}l = \frac{15\cancel{mg}l}{13} + \frac{4l^2k\cancel{mg}}{2l} + \frac{9l\cancel{mg}}{13}$$

cancel the 'l'

$$3\cancel{l} = \frac{15}{13}\cancel{l} + 2\cancel{l}k + \frac{9\cancel{l}}{13}$$

collect like terms

$$2k = 3 - \frac{15}{13} - \frac{9}{13}$$

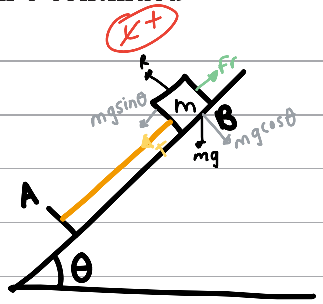
$$\Rightarrow 2k = 15/13$$

$\div 2$ $\div 2$

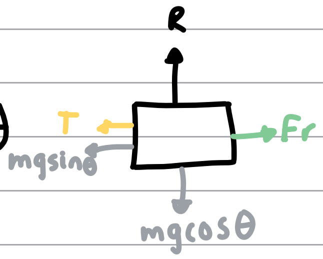
$$\boxed{k = 15/26}$$

(b) now want to focus on drawing a force diagram for when the package is ABOUT TO SLIDE DOWN the plane from B - this direction is important in determining in which direction the friction is acting this time (opposite to motion)

Question 6 continued



turn axis



moving down plane, so R(↙)

- it's an acceleration problem, so from Mech Yr 2 – Newton's 2nd law

$$mg\sin\theta + T - \frac{3}{13}mg = ma$$

↖ from (a)

$T = \frac{\lambda x}{\ell}$

$$\cancel{m}g\left(\tfrac{5}{13}\right) + \frac{\frac{15}{26}\cancel{m}g(2\cancel{\ell})}{\cancel{\ell}} - \frac{3}{13}\cancel{m}g = \cancel{m}a$$

$$\frac{5}{13}g + \frac{15}{13}g - \frac{3}{13}g = a$$

∴ collecting like terms

$$\Rightarrow \boxed{a = \frac{17}{13}g} \ \text{ms}^{-1}$$

Question 6 continued

(Total for Question 6 is 11 marks)

7.

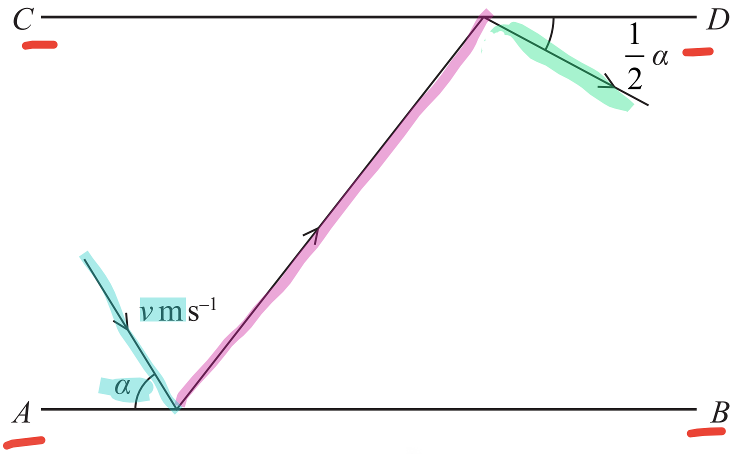


Figure 2

Figure 2 represents the plan view of part of a horizontal floor, where AB and CD represent fixed vertical walls, with AB parallel to CD.

A small ball is projected along the floor towards wall AB. Immediately before hitting wall AB, the ball is moving with speed v m s^{-1} at an angle α to AB, where $0 < \alpha < \frac{\pi}{2}$

The ball hits wall AB and then hits wall CD.

After the impact with wall CD, the ball is moving at angle $\frac{1}{2}\alpha$ to CD.

The coefficient of restitution between the ball and wall AB is $\frac{2}{3}$

The coefficient of restitution between the ball and wall CD is also $\frac{2}{3}$

The floor and the walls are modelled as being smooth. The ball is modelled as a particle.

(a) Show that $\tan\left(\frac{1}{2}\alpha\right) = \frac{1}{3}$

(7)

(b) Find the percentage of the initial kinetic energy of the ball that is lost as a result of the two impacts.

(4)

(a)
now we have an oblique collisions question - useful to know the general idea of the question ; if asked for $\tan(\frac{1}{2}\alpha)$ - must mean that need to find FINAL VELOCITY of the small ball after its 2nd collision - the one with CD

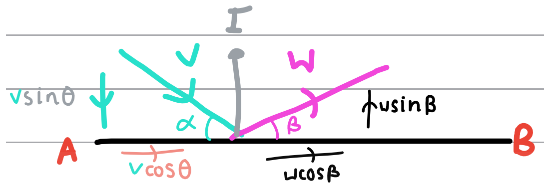
... two main ways to approaching this successive impacts question:

WAY 1: diagrammatically	WAY 2: formulaic
NOTE: this method allows you to just use given 'v' and 'α' variables - no need to work in 'w' as velocity after or 'B' as the angle after	NOTE: in this method - you do have to use intermediate angles and solve resulting equtns simultaneously

P 6 6 5 0 7 A 0 2 4 2 8

Question 7 continued

...first illustrating just the first collision of ball with AB - splitting 'v' into parallel and perpendicular components



...perp. component:

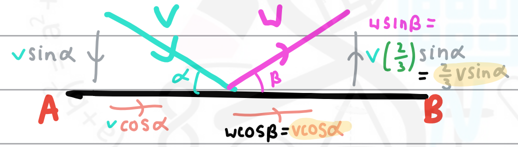
IMPULSE acts perp. to surface of impact ∴ Only perp. component of velocity changes - impacted by NEL (multiply by e)

↳ on diagram

...parallel component:

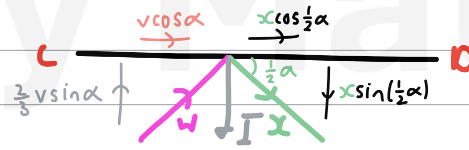
no impact so no change

↳ on diagram



...now second collision - one with CD:

↳ notice velocity before is the same pink 'w' velocity ∴ just copying appropriate components down:

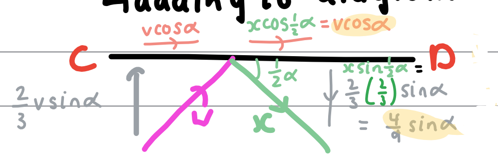


and for the green velocity comps:

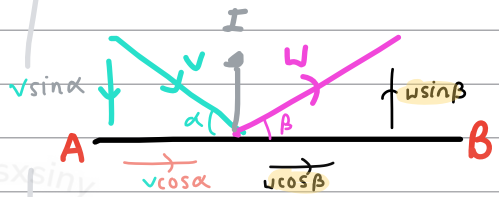
...perpendicular: multiply by $e = 2/3$ (given)

...parallel: remain unchanged

↳ adding to diagram



...first focusing on just first collision of ball and wall - one with AB - split 'v' into its parallel and perpendicular components:



... perp. component:

IMPULSE acts perp. to fixed surface ∴ Only perp. components of velocity change - impacted by NEL (multiply by e)

$\Rightarrow e v\sin\alpha = w\sin\beta$

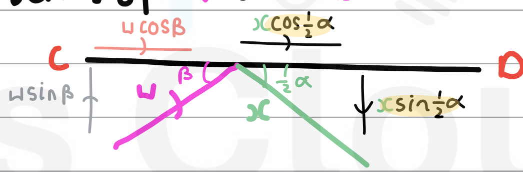
$\Rightarrow \frac{2}{3} v\sin\alpha = w\sin\beta$ - ①

...parallel components:

no impact ∴ no change

$v\cos\beta = w\cos\beta$ - ②

...now focusing on second collision - one with CD - see through alternate angles that same velocity and angle - in terms of 'w' and 'β'



... perpendicular:

NEL rearranged:

$e w\sin\beta = x\sin\frac{1}{2}\alpha$

$\Rightarrow \frac{2}{3} u\sin\beta = x\sin(\frac{1}{2}\alpha)$ - ③

...parallel:

$w\cos\beta = x\cos(\frac{1}{2}\alpha)$ - ④

question asks for equtn in terms of 'α', not 'β'

sub ① into ②

$\frac{2}{3}\left(\frac{2}{3} v\sin\alpha\right) = x\sin(\frac{1}{2}\alpha)$

Turn over

P 6 6 5 0 7 A 0 2 5 2 8

Question 7 continued

$\therefore$ using trig on given $\frac{1}{2}\alpha$ angle

$$\tan\left(\tfrac{1}{2}\alpha\right) = \frac{\frac{4}{9}\cancel{v}\sin\alpha}{\cancel{v}\cos\alpha}$$

$$\boxed{\tan\left(\tfrac{1}{2}\alpha\right) = \tfrac{4}{9}\tan\alpha}$$

$\Rightarrow \frac{4}{9}v\sin\alpha = x\sin\left(\frac{1}{2}\alpha\right)$ - ⑤

sub ② into ④

$v\cos\alpha = x\cos\left(\frac{1}{2}\alpha\right)$ - ⑥

finally for 'tanα': ⑤ ÷ ⑥

$$\tan\left(\tfrac{1}{2}\alpha\right) = \frac{4/9\,\cancel{v}\sin\alpha}{\cancel{v}\cos\alpha}$$

$$\boxed{\therefore \tan\left(\tfrac{1}{2}\alpha\right) = \tfrac{4}{9}\tan\alpha}$$

but because we need a numerical value - need to replace 'tanα' - hint to use tan double angle formula: $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ on $\tan\left(\frac{1}{2}\alpha\right) = \frac{4}{9}\tan\alpha$

let $t = \tan\frac{\alpha}{2}$

$$t = \frac{4}{9}\left(\frac{2t}{1-t^2}\right)$$

$$\Rightarrow \cancel{t} = \frac{8\cancel{t}}{9(1-t^2)} \quad \Rightarrow \quad 1 = \frac{8}{9(1-t^2)}$$

$\times(1-t^2) \quad \times(1-t^2)$

$$(1-t^2) = \frac{8}{9}$$

$\Rightarrow t^2 = 1/9$ so $t = \pm 1/3$

$\tan\frac{\alpha}{2} = \pm 1/3$

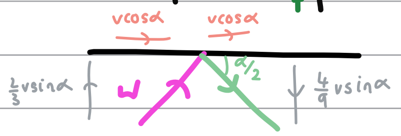
but $0 < \alpha < \pi/2$

$\boxed{\therefore \tan\frac{\alpha}{2} = 1/3}$ as required

(b) following formula for K.E:

$K.E_i = \frac{1}{2}mv^2$

...but for $K.E_f$, use diagram:



...need $\frac{1}{2}m|x|^2$

$$K.E_f = \frac{1}{2}m\left(\sqrt{v^2\cos^2\alpha + \frac{16}{81}v^2\sin^2\alpha}\right)^2$$

need to replace algebraic expressions with numbers - can find tanα

from (a)

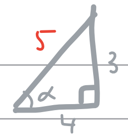
$\tan\left(\frac{1}{2}\alpha\right) = \frac{4}{9}\tan\alpha$

Question 7 continued

$$\frac{1}{3} = \frac{4}{9}\tan\alpha$$

$\div 4/9$ $\quad$ $\div 4/9$

$$\Rightarrow \tan\alpha = 3/4$$



$\sin\alpha = 3/5$
$\cos\alpha = 4/5$

...using trig triangle:

$$\cos^2\alpha = \left(\frac{4}{5}\right)^2 = 16/25$$

$$\sin^2\alpha = (3/5)^2 = 9/25$$

...subbing back into $K.E_f$

$$K.E_f = \frac{1}{2}m\left(\sqrt{v^2\left(\frac{16}{25}\right) + \frac{16}{9}v^2\left(\frac{1}{25}\right)}\right)^2$$

$$= \frac{1}{2}m\left(\frac{16}{25}v^2 + \frac{16}{225}v^2\right)$$

$$\Rightarrow K.E_f = \frac{1}{2}m\left(\frac{32}{45}v^2\right) = \frac{32}{90}mv^2$$

so if $K.E_i = \frac{1}{2}mv^2$

$$K.E_f = \frac{32}{90}mv^2$$

then $K.E_{lost} = K.E_i - K.E_f$

$$= \frac{32}{90}mv^2 - \frac{1}{2}mv^2$$

$$= \frac{13}{90}mv^2$$

which as a % change:

$$\frac{\frac{13}{90}mv^2}{\frac{1}{2}mv^2} \times 100 = 28.88..$$

$$\boxed{= 28.9\%\ (3\text{ s.f})}$$

Question 7 continued

(Total for Question 7 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS